International Journal of Theoretical Physics, Vol. 3, No. 2 (1970), pp. 109–114

Realisations of the Representations of para-Fermi Algebra in Fock Space of Bose Operators: Part I

K. KADEMOVA†

International Centre for Theoretical Physics, Trieste, Italy

Received: 23 *October* 1969

Abstract

Using the matrix realisations of para-Fermi operators we find isomorphic mappings with respect to the Green product of the para-Fermi algebra into second-order polynomials of creation and annihilation para-Bose operators with arbitrary order of parastatistics. In the Fock space $\mathscr{H}_2{}^1$ of two Bose operators all the irreducible representations of the para-Fermi algebra are realised. The spaces of n-particle Bose states $n = 1, 2, \ldots$, from which $\mathscr{H}_2{}^1$ is constructed as a direct sum, can be interpreted as spaces of para-Fermi states of para-statistics n.

1. *Introduction*

The properties of para-Fermi and para-Bose operators introduced by Green (1953) and their possible physical applications have been carefully studied by Greenberg and other authors in a series of papers (Greenberg, 1964, 1965a; Greenberg & Messiah, 1965b, c; Volkov, 1959; Govorkov, 1966).

In the present paper we show in a constructive way that para-Fermi algebra operators can be expressed as functions of creation and annihilation para-Bose operators. This gives us a possibility of giving some new physical interpretation of para-Fermi field operators.

Making use of the matrix realisations of the para-Fermi algebra given by Green, we find isomorphic mappings of para-Fermi algebra into second-order polynomials of creation and annihilation para-Bose operators (Section 3). In Section 4, in the Fock space $\mathscr{H}_2{}^1$ of two Bose operators, all the irreducible representations of para-Fermi algebra are found. A consistent interpretation can be given of the $n + 1$, $n = 1, 2, \ldots$, dimensional subspace of $\mathscr{H}_2{}^1$ spanned by n-particle states of Bose operators, which is invariant under the transformations induced by the para-Fermi algebra, as a space spanned on the states of para-Fermi operators with parastatistics n.

† On leave of absence from the Institute of Physics, Bulgarian Academy of Sciences, Sofia, Bulgaria.

2. *Notations and Basic Notions*

The Green algebra we define as a vector space $\mathbf{G}_{2n}$ over the field of the complex numbers $\mathbf{C}$ spanned by $2n$ parafield operators $\overset{+}{a}_i$, a_i, $i = 1, 2, \ldots, n$, for which the Green product is defined as a double bracket†

$$[\tfrac{1}{2}[\overset{+}{a}_i, a_j]_{\pm}, \overset{+}{a}_k]_- = \delta_{kj}\overset{+}{a}_i \tag{2.1}$$

$$[\tfrac{1}{2}[\overset{+}{a}_i, \overset{+}{a}_j]_{\pm}, \overset{+}{a}_k]_- = 0 \tag{2.2}$$

where $i, j, k = 1, 2, \ldots, n$.

All the other commutation relations are obtained from (2.1) and (2.2) using conjugation and Jacobi identity.

The Green algebra has different realisations (Green, 1953), each one characterised by an integer positive number $p = 1, 2, \ldots$, the order of parastatistics of the algebra.

Further, we shall use the notations $\overset{+}{f_i}{}^p, f_i{}^p$ for para-Fermi operators with parastatistics p; $\overset{+}{b_i}{}^p$, $b_i{}^p$ for the corresponding para-Bose operators and $\mathbf{F}^p_{2n}$, $\mathbf{B}^p_{2n}$ for their Green algebras. The case $p = 1$ corresponds to the usual Fermi and Bose operators and their algebras.

In the case $n = 1$ Green has found (Green, 1953) simple irreducible matrix realisations of the para-Fermi algebra $\mathbf{F}^p$ of the form

$$(F^p)_{mn} = \delta_{mn-1}[m(p - m + 1)]^{1/2}, \qquad 1 \leqslant m, n \leqslant p + 1$$

$$(\overset{+}{F^p})_{mn} = \mathbf{F}^p_{nm} \tag{2.3}$$

For $n > 1$ no simple irreducible matrix representation is found. In this case the reducible representations are constructed through Green's ansatz.

Since for $n > 1$ no additional complications arise, for simplicity we shall construct the isomorphisms of the para-Fermi algebra for the case $n = 1$.

As regards the representations of the Green algebras, Greenberg has proved (Greenberg & Messiah, 1965c) that all the irreducible representations in a Hilbert space, which has a unique no particle state $|0\rangle$, satisfying

$$a_i|0\rangle = 0, \qquad i = 1, \ldots, n, \tag{2.4}$$

also satisfy

$$a_l\overset{+}{a}_k|0\rangle = p\delta_{lk}|0\rangle, \qquad l, k = 1, \ldots, n, \tag{2.5}$$

with p a positive integer, and are characterised by (2.4) and (2.5) up to unitary equivalence.

For every order of parastatistics p, particle number operators are introduced as, e.g.,

$$N_k = \tfrac{1}{2}([\overset{+}{a}_k, a_k]_{\pm} \mp p), \qquad k = 1, \ldots, n, \tag{2.6}$$

† The cases of para-Bose and para-Fermi algebras are considered simultaneously. The double bracket is defined by the sign '+' for the para-Bose algebra case and by '−' for the para-Fermi algebra case.

which, in the case $p = 1$, coincide with the operator $N_k = \overset{+}{a}_k a_k$. The Fock space $\mathscr{H}_n{}^p$ of para-Bose algebra B^q_{2n} is spanned on the vectors (Kademova, 1969)

$$|\alpha_{i_\epsilon}\rangle = \prod_{i,\epsilon} (\overset{+}{b}_i{}^p)^{\alpha_{i\epsilon}}|0\rangle \tag{2.7}$$

where α_{i_ϵ} are positive integers, the index ϵ indicates the place in the ordered product in which the operator $\overset{+}{b}_i{}^p$ in power α_{i_ϵ} stands.

3. *Isomorphism of the para-Fermi Algebra into Second-order Polynomials of para-Bose Operators*

For constructing the isomorphism of the para-Fermi algebra F^p into the second-order polynomials of para-Bose operators with parastatistics q, we consider first the free associative algebra Φ^q_{2p+2} generated by $2p + 2$ (non-commuting) indeterminants $\overset{+}{b}_i{}^q$, $b_i{}^q$, $i = 1, \ldots, p + 1$, (q is an arbitrary fixed integer) over the field of the complex numbers C. The brackets $[b_i{}^q, \overset{+}{b}_j{}^q]_+$, $[b_i{}^q, b_j{}^q]_+$, $[\overset{+}{b}_i{}^q, \overset{+}{b}_j{}^q]_+$ in Φ^q_{2p+2} are defined in the natural way. Φ^q_{2p+2} can be considered as a space spanned on all tensorial products of the operators $\overset{+}{b}_i{}^q$, $b_i{}^q$. Then we define the enveloping algebra $\epsilon^q_{2p+2} = \Phi^q_{2p+2}/\mathscr{I}(\mathrm{B}^q_{2p+2})$, where $\mathscr{I}(\mathrm{B}^q_{2p+2})$ is the two-sided ideal generated by the commutation relations (2.1) and (2.2) which correspond to the para-Bose case. We shall consider the subspace $\epsilon^{q(2)}_{2p+2} = \{\overset{+}{b}_i{}^q \overset{+}{b}_j{}^q, b_i{}^q b_j{}^q, \overset{+}{b}_i{}^q b_j{}^q, b_i{}^q \overset{+}{b}_j{}^q; i, j = 1, \ldots, p + 1\}$ of the space ϵ^q_{2p+2}. We shall define an isomorphism with respect to the Green product of the para-Fermi algebra F^p into $\epsilon^{q(2)}_{2p+2}$.

Theorem

For F^p arbitrary para-Fermi algebra, the mapping

$$i^p_q: \mathscr{F}_q{}^p = \sum_{i,j=1}^{p+1} (F^p)_{ij}\tfrac{1}{2}[\overset{+}{b}_i{}^q, b_j{}^q]_+ \tag{3.1}$$

is an isomorphism of F^p into $\epsilon^{q(2)}_{2p+2}$.

Proof: From Kademova (1969, proposition 1) it follows directly that†

$$[\tfrac{1}{2}[\overset{+}{\mathscr{F}}_q{}^p, \mathscr{F}_q{}^p]_-, \overset{+}{\mathscr{F}}_q{}^p]_- = ([\tfrac{1}{2}[\overset{+}{F}{}^p, F^p]_-, \overset{+}{F}{}^p]_-)_{ij}\tfrac{1}{2}[\overset{+}{b}_i{}^q, b_j{}^q]_+$$
$$= (\overset{+}{F}{}^p)_{ij}\tfrac{1}{2}[\overset{+}{b}_i, b_j]_+ = \overset{+}{\mathscr{F}}_q{}^p$$

and

$$[\tfrac{1}{2}[\mathscr{F}_q{}^p, \mathscr{F}_q{}^p]_-, \mathscr{F}_q{}^p]_- = ([\tfrac{1}{2}[F^p, F^p]_-, F^p]_-)_{ij}\tfrac{1}{2}[\overset{+}{b}_i{}^q, b_j{}^q]_+ = 0$$

† We adopt the summation convention over repeated low indices.

From this, and from the fact that the matrices (2.3) form a faithful representation of the para-Fermi algebra for every fixed p, it follows that (3.1) is an isomorphic mapping with respect to the Green product of the para-Fermi algebra $\mathbf{F}^p$ into $\epsilon_{2p+2}^{q(2)}$.

The isomorphism we have constructed is only with respect to the Green product. It does not preserve the particular commutation relations for every para-Fermi algebra with fixed order of parastatistics p. We shall denote by $\mathscr{F}_q{}^p$ the para-Fermi algebra isomorphic to $\mathbf{F}^p$ through the isomorphism i_q^p.

4. *Realisations of the Representations of the para-Fermi Algebra* $\mathbf{F}^1$ *in the Fock Space* $\mathscr{H}_2{}^1$ *of the para-Bose Algebra* $\mathbf{B}_4{}^1$

We shall restrict ourselves to realisations of the representations of $\mathbf{F}^1$ in the space $\mathscr{H}_2{}^1$, since due to the Greenberg theorem we shall show that all the unitary non-equivalent representations of the para-Fermi algebra will be constructed in this way.

Using formula (2.3) for the parastatistics case $p = 1$, i.e., for the Fermi statistics case, we construct the para-Fermi algebra $\mathscr{F}_1{}^1$ isomorphic to $\mathbf{F}^1$ by means of Bose operators in the form (3.1)

$$\mathscr{F}_1{}^1 = \tfrac{1}{2}[\overset{+}{b}_1, b_2]_+ = \overset{+}{b}_1 b_2$$
$$\overset{+}{\mathscr{F}}_1{}^1 = \tfrac{1}{2}[\overset{+}{b}_2, b_1]_+ = \overset{+}{b}_2 b_1 \tag{4.1}$$

The Fock space $\mathscr{H}_2{}^1$ of these Bose operators $\overset{+}{b}_1, \overset{+}{b}_2$ is spanned on the vectors

$$|\alpha_1, \alpha_2\rangle = (\overset{+}{b}_2)^{\alpha_2}(\overset{+}{b}_1)^{\alpha_2}|0\rangle \tag{4.2}$$

where $|0\rangle$ is the Bose vacuum state

$$b_i|0\rangle = 0, \qquad i = 1, 2$$

Let us consider first the two-dimensional subspace H_1 of the space $\mathscr{H}_2{}^1$ spanned on the single-particle Bose states $|1,0\rangle$, $|0,1\rangle$ and the transformations induced by the algebra $\mathscr{F}_1{}^1$. From (4.1) and (4.2) follows:

$$\mathscr{F}_1{}^1|1,0\rangle = 0$$
$$\mathscr{F}_1{}^1|0,1\rangle = |1,0\rangle$$
$$\overset{+}{\mathscr{F}}_1{}^1|1,0\rangle = |0,1\rangle$$
$$\overset{+}{\mathscr{F}}_1{}^1|0,1\rangle = (\overset{+}{\mathscr{F}}_1{}^1)^2|1,0\rangle = 0$$

This means that the subspace H_1 is invariant under these transformations.

Moreover, from this it follows that the single-particle Bose state $|1,0\rangle$ can be considered as a vacuum $|0\rangle_{\mathscr{F}_1{}^1}$, and the other single-particle Bose state $|0, 1\rangle$ as a single-particle state $|1\rangle_{\mathscr{F}_1{}^1}$ for $\mathscr{F}_1{}^1$ algebra. This interpretation is consistent with the result received after applying the number-particle operator (2.6) to these vectors.

One can directly check that the operators $\mathscr{F}_1{}^1$ and $\overset{+}{\mathscr{F}}_1{}^1$ satisfy, in this subspace, Fermi commutation relations. This also follows straight from

$$\mathscr{F}_1{}^1\overset{+}{\mathscr{F}}_1{}^1|0\rangle_{\mathscr{F}_1{}^1} = \mathscr{F}_1{}^1\overset{+}{\mathscr{F}}_1{}^1|1, 0\rangle = |1, 0\rangle = |0\rangle_{\mathscr{F}_1{}^1}$$

so that $p = 1$. So in the subspace H_1 the transformations induced by $\mathscr{F}_1{}^1$ form the usual Fermi algebra.

Now we consider the subspace H_n spanned on the n-particle Bose states†

$$|\alpha_1, \alpha_2\rangle = \frac{\overset{+}{b}{}_2^{\alpha_2}\overset{+}{b}{}_1^{\alpha_1}}{\sqrt{(\alpha_1!\,\alpha_2!)}}|0\rangle$$

where $\alpha_1 + \alpha_2 = n$.

Then the transformations induced by $\mathscr{F}_1{}^1$ in this subspace are:

$$\mathscr{F}_1{}^1|\alpha_1, \alpha_2\rangle = \sqrt{[(\alpha_1 + 1)\alpha_2]}|\alpha_1 + 1, \alpha_2 - 1\rangle$$

$$\overset{+}{\mathscr{F}}_1{}^1|\alpha_1, \alpha_2\rangle = \sqrt{[\alpha_1(\alpha_2 + 1)]}|\alpha_1 - 1, \alpha_2 + 1\rangle$$

which shows again that the space H_n is invariant under this transformation.

The $n + 1$ vectors $|\alpha_1, \alpha_2\rangle$ can be considered as α_2-particle states

$$|\alpha_2\rangle_{\mathscr{F}_1{}^1} = |\alpha_1, \alpha_2\rangle \tag{4.3}$$

for the algebra $\mathscr{F}_1{}^1$. Again applying the number operator (2.6)

$$N|\alpha_1, \alpha_2\rangle = \alpha_2|\alpha_1, \alpha_2\rangle$$

we see that such an interpretation is possible. Since

$$\mathscr{F}_1{}^1\overset{+}{\mathscr{F}}_1{}^1|n, 0\rangle = n|n, 0\rangle$$

it follows that the para-Fermi algebra of the transformations in H_n induced by $\mathscr{F}_1{}^1$ corresponds to parastatistics n.

In this way we have proved that in the Fock space of two Bose operators, representations of the para-Fermi algebra $\mathscr{F}_1{}^1$ exist such that every subspace H_n, spanned on n-particle states of Bose operators $n = 1, 2, \ldots$, is invariant under the transformations induced by $\mathscr{F}_1{}^1$. In this subspace the para-Fermi algebra induced by $\mathscr{F}_1{}^1$ corresponds to parastatistics n.

Moreover, the n-particle Bose states can be regarded as para-Fermi particle states with parastatistics n. So the Fock space of two Bose operators $\mathscr{H}_2{}^1$ can be considered as a direct sum of spaces H_p spanned on the states of para-Fermi operators with parastatistics $p = 1, 2, \ldots$. So we found all unitary non-equivalent irreducible representations of the para-Fermi algebra in the space $\mathscr{H}_2{}^1$.

† We introduced a normalisation factor for convenience.

Acknowledgements

The author is grateful to Professor A. Kálnay for valuable discussions and for critical reading of the manuscript.

She is also indebted to Professors Abdus Salam and P. Budini and the International Atomic Energy Agency for the hospitality kindly extended to her at the International Centre for Theoretical Physics, Trieste.

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